Journal of Statistical Physics, Vol. 66, Nos. 5/6, 1992

Invariance Principle for the Stochastic Lorentz Lattice Gas

F. den Hollander,[1] J. Naudts,[2] and F. Redig[3]

Received June 18, 1991

We prove scaling to nondegenerate Brownian motion for the path of a test particle in the stochastic Lorentz lattice gas on $\mathbb{Z}^d$ under a weak ergodicity assumption on the scatterer distribution. We prove that recurrence holds almost surely in $d \leqslant 2$. Transience in $d \geqslant 3$ remains open.

KEY WORDS: Nonreversibility; scatterer time scale; invariance principle; Brownian motion; stationary ergodic and reversible measures; connectedness; Dirichlet principle; effective medium.

1. INTRODUCTION

The motion of a test particle in the discrete-time stochastic Lorentz lattice gas can be described as follows. Consider the d-dimensional lattice $\mathbb{Z}^d$, $d \geqslant 1$. Scatterers are placed randomly on the sites according to some probability measure μ that is stationary and ergodic under translations. A test particle starts at the origin with unit velocity in one of the $2d$ coordinate directions and moves in a straight line, one step per unit of time, until it hits a first scatterer. There the velocity randomly changes direction and the particle continues to move in a straight line until it hits a second scatterer, and so on.

In the continuous-time version of this model the particle jumps on the event times of a mean-one Poisson process, which is independent of the position of the particle and of the scatterer configuration.

In this paper we prove that the path of the particle scales to nondegenerate Brownian motion both for the discrete- and for the continuous-

[1] Mathematical Institute, University of Utrecht, NL-3508 TA Utrecht, The Netherlands.
[2] Department of Physics, Antwerpen University, B-2610 Antwerpen, Belgium.
[3] Aspirant NFWO, Antwerpen University, B-2610 Antwerpen, Belgium.

822/66/5-6-26

time version.[4] The main difficulty is that the environment process (i.e., the process of scatterer configurations as seen from the position of the particle) is non-Markov. The joint process of environment and velocity is Markov, but fails to be reversible. However, if the particle starts on a scatterer (with random initial velocity) and if it is observed only when hitting a scatterer, then the imbedded environment process is Markov and reversible w.r.t. μ_0, the probability measure obtained from μ by conditioning on the origin to be a scatterer (i.e., the particle starts on a scatterer with random initial velocity). In other words, the environment process is *Markov and reversible along the random time scale of scatterer hitting times* (the Markov property relies on the assumption of uniform scattering, i.e., scattering with equal probability in each direction). In order to ensure that the latter process is ergodic, we need a certain notion of *connectedness*. This means that the test particle can reach any scatterer from any other (i.e., the imbedded jump process is irreducible). We are then in a situation where we can apply a theorem by DeMasi *et al.*[(1)] to obtain scaling to Brownian motion (invariance principle) along the random time scale of scatterer hitting times.

At that stage three problems remain to be solved. First, to get back to the full time scale, we use a random-time-change argument. Second, to deal with the starting measure μ (i.e., the situation where the origin is not conditioned to be a scatterer and the initial velocity of the particle is arbitrary), we prove that when the particle hits the first scatterer the environment it sees is distributed according to a measure that is absolutely continuous w.r.t. μ_0. The latter implies that the scaling carries over with the same diffusion matrix. Third, to prove nondegeneracy of the limiting Brownian motion, we exhibit lower and upper bounds for the diffusion matrix.

Finally, we prove recurrence in dimension $d=2$. This property cannot be derived from the invariance principle, because the indicator of returning infinitely often to the origin is not a continuous function on path space. Recurrence is obtained from a comparison inequality which shows that the path of the test particle in the random medium is in a certain sense "more recurrent" than in the average (effective) medium. The proof of the inequality is based on the Dirichlet principle. Transience in $d \geqslant 3$ remains open.

The paper is organized as follows. In Section 2 we introduce a general model where the test particle jumps between scatterers at a rate depending on the interscatterer distance. We prove an invariance principle for the rescaled position of the test particle. In Section 3 we show how to apply the

[4] A referee has pointed out to us that the same result is announced without proof in Varadhan.[(9)] A proof has never been published (S. R. S. Varadhan, private communication).

latter result in order to prove the invariance principle for the Stochastic Lorentz lattice gas on the scatterer hitting time scale and for μ_0. We then show how to obtain the invariance principle on the full time scale and for the nonconditioned measure μ. In Section 4 we prove that the diffusion matrix is nondegenerate. We shall see (as was already noticed by other authors) that the diffusion constant differs considerably from the Boltzmann value. In $d=1$ the diffusion constant equals the mean interscatterer distance. In $d \geqslant 2$ we exhibit classical bounds: a lower bound (the one-dimensional diffusion constant) and an upper bound (the Boltzmann value). In Section 5 we deal with the problem of recurrence.

2. A CONTINUOUS-TIME RANDOM WALK ON SCATTERERS

We introduce a continuous-time random walk on scatterers with the following properties:

(i) Jumps are parallel to the coordinate directions and occur only between neighboring scatterers (i.e., the walk cannot jump over a scatterer).

(ii) The jump rates depend symmetrically on the jump vector.

The main point of this section is to establish a sufficient condition for ergodicity of the environment process. This is needed for the invariance principle.

2.1. Notations

The scatterer configuration is denoted by a mapping η: $\mathbb{Z}^d \to \{0, 1\}$ with $\eta(x) = 1$ if x is a scatterer and $\eta(x) = 0$ if not. Ω denotes the set of all scatterer configurations, and

$$\Omega_0 = \{\eta \in \Omega: \eta(0) = 1\}$$

plays the role of state space of the environment process (EP). The condition $\eta(0) = 1$ reflects the fact that the random walk is confined to scatterer positions. $S_d \subset \mathbb{Z}^d$ denotes the set of $2d$ unit vectors. For $\eta \in \Omega$ and $e \in S_d$

$$u(e, \eta) = |u(e, \eta)|\, e$$

$$|u(e, \eta)| = \inf\{k > 0: \eta(ke) = 1\}$$

i.e., $u(e, \eta)$ is the position of the scatterer nearest to the origin in the direction of e. For $\eta \in \Omega$ and $a \in \mathbb{Z}^d$, $\tau_a \eta$ denotes the scatterer configuration shifted by a, i.e., $\tau_a \eta(x) = \eta(x + a)$.

2.2. Definition of the Random Walk

Let $\psi_e: \mathbb{N} \to (0, \infty)$ be bounded functions indexed by $e \in S_d$ such that $\psi_e = \psi_{-e}$ and put $\psi_e(\infty) = 0$. Given $\eta \in \Omega_0$, the random walk is defined as the continuous-time Markov process $\{X_t: t \geqslant 0\}$, on the set of scatterer positions, with $X_0 = 0$ and with transition probabilities $P^\eta(X_{t+h} = y \mid X_t = x) = h\psi_e(|x-y|) + o(h)$ $(h \to 0)$ for $x, y \in \eta^{-1}\{1\}$ and $e \in S_d$ such that $y - x = u(e, \tau_x \eta)$, and $P^\eta(X_{t+h} = x \mid X_t = x) = 1 - \sum_{y \neq x} P^\eta(X_{t+h} = y \mid X_t = x)$.

The corresponding EP $\{\eta_t: t \geqslant 0\}$ on Ω_0 is defined by $\eta_t = \tau_{X_t} \eta$. Let $B(\Omega_0)$ denote the space of bounded Borel-measurable functions on Ω_0. Then the generator L of the EP is the linear operator on $B(\Omega_0)$ given by

$$
\begin{aligned}
Lf(\eta) &\equiv \lim_{t \downarrow 0} \frac{1}{t} E^\eta[f(\eta_t) - f(\eta_0)] \\
&= \sum_{e \in S_d} \psi_e(|u(e, \eta)|)[f(\tau_{u(e,\eta)}\eta) - f(\eta)]
\end{aligned} \tag{2.1}
$$

E^η is expectation w.r.t. P^η, the Markov measure on path space given η. Let $\{S(t): t \geqslant 0\}$ denote the semigroup of the EP.

2.3. Stationary, Ergodic, and Reversible Measures

For $\eta \in \Omega$ and $x, y \in \eta^{-1}\{1\}$, define a path from x to y along scatterers to be a set $\{x_1, ..., x_n\} \subset \eta^{-1}\{1\}$ such that $x_1 = x$, $x_n = y$, and $x_{m+1} - x_m = u(e_m, \tau_{x_m}\eta)$ for some $e_m \in S_d$, $m = 1, ..., n-1$. We say that x and y are *connected* in η, and write $x \sim y$, if there exists a path from x to y along scatterers.

Definition. A configuration η is connected if $x \sim y$ for all $x, y \in \eta^{-1}\{1\}$. A probability measure μ on Ω is connected if η is connected μ-a.s.

The following set of conditions will be needed later.

Conditions (C).

(i) μ is stationary and ergodic under translations.

(ii) μ is connected.

(iii) $\mu(\eta(0) = 1) > 0$.

Theorem 2.1. Let μ be a probability measure on Ω satisfying (C). Then $\mu_0(\cdot) = \mu(\cdot \mid \eta(0) = 1)$ is stationary, ergodic, and reversible for the EP.

Proof. The stationarity and reversibility are straightforward. Indeed, one easily checks by explicit computation from (2.1) that for all $f, g \in B(\Omega_0)$

$$\int Lf \, d\mu_0 = 0$$

$$\int f(Lg) \, d\mu_0 = \int (Lf) \, g \, d\mu_0$$

using the stationarity of μ under translations and the assumption $\psi_e = \psi_{-e}$ on the rates.

The ergodicity is more subtle and depends on the connectedness of μ. Suppose that μ_0 is not ergodic. Then there exists a measurable set $E \subset \Omega_0$, with $0 < \mu_0(E) < 1$, which is invariant under the semigroup, i.e.,

$$S(t) 1_E = 1_E \qquad \text{for all} \quad t \geqslant 0$$

(cf. ref. 8). The complement $E^c = \Omega_0 \backslash E$ has the same properties. From the ergodicity of μ under translations it follows that

$$\mu\left(\bigcup_{x \in Z^d} \tau_x E\right) = \mu\left(\bigcup_{x \in Z^d} \tau_x E^c\right) = 1$$

Hence μ-a.s. for all η there exist $x, y \in \mathbb{Z}^d$ such that $\tau_x \eta \in E$ and $\tau_y \eta \in E^c$. Now pick $0 < t_1 < t_2 < \infty$ independent of x and y. Because 0, x, and y are μ_0-a.s. connected and because the jump rates are positive, it follows that

$$\int P^\eta(X_{t_1} = x, X_{t_2} = y) \, \mu_0(d\eta) > 0$$

But $X_{t_1} = x$ and $X_{t_2} = y$ imply $\eta_{t_1} = \tau_x \eta \in E$ and $\eta_{t_2} = \tau_y \eta \in E^c$, whereas by the invariance of E, $\eta_{t_1} \in E$ implies $\eta_{t_2} \in E$. This is a contradiction. ∎

2.4. Connectedness

In $d = 1$ every measure is connected. The following theorem gives a sufficient condition for $d \geqslant 2$.

Theorem 2.2. Let μ be a probability measure on Ω satisfying:

(i) μ is stationary and ergodic under translations in each of the coordinate directions.

(ii) $GCD\{k > 0 \colon \mu(\eta(0) = \eta(ke) = 1) > 0\} = 1$ for all $e \in S_d$.

Then μ is connected.

Proof. For $x \in \mathbb{Z}^d$ and $e \in S_d$ define the line

$$R_{x,e} = \{x + ke: k \in \mathbb{Z}\}$$

All scatterers on a line are connected. We say that two sets $A, B \subset \mathbb{Z}^d$ are connected, and write $A \sim B$, if all scatterers in A are connected to all scatterers in B. Our first observation is that for all $x \in \mathbb{Z}^d$, e, $e' \in S_d$, and $k > 0$,

$$\eta(x + le') = \eta(x + ke + le') \text{ for some } l > 0 \Rightarrow R_{x,e'} \sim R_{x+ke,e'}$$

i.e., two parallel coordinate lines in a coordinate plane can be connected via a single pair of scatterers. Now define

$$A_e = \{k > 0: \mu(\eta(0) = \eta(ke) = 1) > 0\}$$

Then by assumption (i) the l.h.s. of the above implication holds μ-a.s. for all $x \in \mathbb{Z}^d$, e, $e' \in S_d$, and $k \in A_e$. By assumption (ii) we therefore have

$$GCD\{k > 0: R_{x,e'} \sim R_{x+ke,e'}\} = 1 \quad \mu\text{-a.s.}$$

Our second observation is that from the transitivity of the relation $\sim$ and the stationarity of μ under translations it now follows that

$$R_{x,e'} \sim R_{x+ke,e'} \quad \mu\text{-a.s.} \quad \text{for all} \quad k > 0$$

Hence η is connected. ∎

Remark. The assumptions in Theorem 2.2 are fairly optimal. One easily checks that (ii) is also necessary under (i).

2.5. Invariance Principle

Under the conditions of Theorem 2.1, we are in a situation to apply the invariance principle (IP) for reversible Markov processes (see ref. 1, Theorem 2.2; one can assume without restriction that configurations are not periodic under translations; then X_t is an antisymmetric function of the environment process). This yields:

Theorem 2.3. Assume (C) and $E_{\mu_0}(|u(e, \eta)|^2 \psi_e(|u(e, \eta)|)) < \infty$ for all $e \in S_d$. Then as $\varepsilon \to 0$ the processes $\{\varepsilon X_{\varepsilon^{-2}t}: t \geq 0\}$ converge weakly in μ_0-measure to a Wiener process $\{W_D(t): t \geq 0\}$. The diffusion matrix D is given by

$$D_{ij} = (D_i^s - D_i^d)\delta_{ij}$$

where

$$D_i^s = \sum_{e=\pm e_i} E_{\mu_0}(|u(e,\eta)|^2 \psi_e(|u(e,\eta)|))$$
$$D_i^d = 2\int_0^\infty dt\, E_{\mu_0}(\Phi_i(\eta)\, S(t)\, \Phi_i(\eta)) \tag{2.2}$$

with

$$\Phi_i(\eta) = \sum_{e=\pm e_i} \psi_e(|u(e,\eta)|)\, u(e,\eta)$$

(e_i, $i=1,\ldots,d$, are the positive unit vectors in the ith coordinate direction).

Remark. D_i^s and D_i^d are called the "static," resp. "dynamic" part of the diffusion constant in the ith coordinate direction. The former is the diffusion constant of the "effective medium" process and is explicitly calculable. The latter depends in a more intricate way on the randomness of the medium and is usually not explicitly calculable. Note that the second assumption is sufficient to ensure the quadratic integrability of X_t and of Φ.

3. THE STOCHASTIC LORENTZ LATTICE GAS

In this section we define the motion of a test particle in the discrete-time stochastic Lorentz lattice gas. First we show that by looking on the scatterer hitting time scale the EP becomes a process of the type discussed in Section 2. Theorem 2.3 will then give us the IP on the random time scale. Next we apply a random-time-change argument to transfer the IP to the full time scale. Finally we show how to extend the IP to the situation where the origin is not conditioned to be a scatterer and the initial velocity of the particle is arbitrary.

3.1. Definition of the Model

The test particle in the discrete-time stochastic Lorentz lattice gas is described by a position X_n and a velocity v_n. Given $\eta \in \Omega$, the state space of the test particle is $Z^d \times S_d$. The EP $\{\tau_{X_n}\eta\colon n \geqslant 0\}$ is not Markovian. However, by adding the velocity and defining the EP as

$$\{(\tau_{X_n}\eta, v_n)\colon n \geqslant 0\}$$

we obtain a Markov process on $\Omega \times S_d$. The transition operator P of this EP is defined on $B(\Omega \times S_d)$ by

$$Pf(\eta, v) = [1-\eta(0)]\, f(\tau_v\eta, v) + \eta(0)\frac{1}{2d}\sum_{e \in S_d} f(\tau_e\eta, e) \tag{3.1}$$

Any measure of the type $\mu \otimes \lambda$, with μ on Ω stationary under translations and λ on S_d the uniform measure, is invariant under P. However, such measures are in general not reversible.

3.2. The Scatterer Time Scale

Suppose that $\eta(0) = 1$. Let $\{T_n: n \geqslant 0\}$ denote the successive times at which the particle hits a scatterer, i.e.,

$$T_0 = 0$$
$$T_{n+1} = \inf\{k > T_n : \eta(X_k) = 1\}, \qquad n \geqslant 0$$

The first marginal of the imbedded EP $\{(\tau_{X_{T_n}}\eta, v_{T_n}): n \geqslant 0\}$ is $\{\tau_{X_{T_n}}\eta: n \geqslant 0\}$ and is a Markov process with transition operator P' on $B(\Omega_0)$ given by

$$P'f(\eta) = \frac{1}{2d} \sum_{e \in S_d} f(\tau_{u(e,\eta)}\eta) \tag{3.2}$$

To the discrete-time imbedded EP there corresponds a continuous-time version with generator $L = P' - I$. It is clear that the latter is a process of the type introduced in Section 2, namely, with $\psi_e \equiv 1/2d$ for all $e \in S_d$ [recall (2.1)]. In particular, Theorem 2.1 shows that it is stationary, ergodic, and reversible w.r.t. $\mu_0(\cdot) = \mu(\cdot \mid \eta(0) = 1)$ when μ satisfies (C). It follows easily that the discrete-time imbedded EP shares the same properties.

Therefore we obtain the following analogue of Theorem 2.3 (see ref. 1, Theorem 2.1).

Theorem 3.1. Assume (C) and $0 < E_{\mu_0}|u(e, \eta)|^2 < \infty$ for all $e \in S_d$. Then as $n \to \infty$ the processes $\{n^{-1/2}X_{T_{[nt]}}: t \geqslant 0\}$ converge weakly in μ_0-measure to a Wiener process $\{W_D(t): t \geqslant 0\}$. The diffusion matrix is given by

$$D_{ij} = (D_i^s - D_i^d)\delta_{ij}$$

where

$$\begin{aligned} D_i^s &= \frac{1}{2d} \sum_{e = \pm e_i} E_{\mu_0}|u(e, \eta)|^2 \\ D_i^d &= 2 \sum_{n=0}^{\infty} E_{\mu_0}(\Phi_i(\eta)\, P'^n \Phi_i(\eta)) \end{aligned} \tag{3.3}$$

with

$$\Phi_i(\eta) = \frac{1}{2d} \sum_{e = \pm e_i} u(e, \eta)$$

3.3. Invariance Principle on the Full Time Scale for μ_0

The process $\{X_{T_n}: n \geqslant 0\}$ has stationary and ergodic increments if the scatterers are distributed according to μ_0. Since

$$T_n = \sum_{k=1}^{n} |X_{T_k} - X_{T_{k-1}}|$$

the Birkhoff ergodic theorem implies the following result.

Lemma 3.2. $\lim_{n \to \infty} n^{-1} T_n = \alpha^{-1}$ a.s., where $\alpha^{-1} = E_{\mu_0}(|X_{T_1}|) = E_{\mu_0}(T_1)$.

The above property of the time scale T_n will be crucial for the random-time-change argument given below. The following is an intermediate result on the full time scale, namely we replace $X_{T_{[nt]}}$ in Theorem 3.1 by $X_{[T_n t]}$.

Lemma 3.3. Assume (C) and $E_{\mu_0} |u(e, \eta)|^{2+\delta} < \infty$ for some $\delta > 0$ and for all $e \in S_d$. Then as $n \to \infty$ the processes $\{n^{-1/2} X_{[T_n t]}: t \geqslant 0\}$ converge weakly in μ_0-measure to the same Wiener process as in Theorem 3.1.

Proof. Let

$$\xi(n) = \sup\{T_m: T_m \leqslant n\}$$

denote the last hitting time of a scatterer prior to time n. Let $\psi_n(t)$ be defined by

$$\xi([T_n t]) = T_{[n\psi_n(t)]}$$

By Lemma 3.2 we have, uniformly on compact intervals of time,

$$\lim_{n \to \infty} \frac{[T_n t]}{T_{[nt]}} = 1 \quad \text{a.s.}$$

and hence

$$\lim_{n \to \infty} \psi_n(t) = t \quad \text{a.s.}$$

Now apply the random-time-change theorem of Billingsley (ref. 2, Chapter 3, Section 3) to obtain from Theorem 3.1 that $\{n^{-1/2} X_{\xi([T_n t])}: t \geqslant 0\}$ converges weakly in μ_0-measure to $\{W_D(t): t \geqslant 0\}$. Next let $Y_n = X_n - X_{n-1}$. Since

$$X_{[T_n t]} = X_{\xi([T_n t])} + \sum_{m=\xi([T_n t])+1}^{[T_n t]} Y_m$$

the proof of the lemma will be complete once we show that uniformly on compacts

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{m=\xi([T_n t])+1}^{[T_n t]} Y_m = 0 \qquad \text{in probability}$$

Indeed, we have the estimates

$$\left| \sum_{m=\xi([T_n t])+1}^{[T_n t]} Y_m \right| \leqslant T_{[n\psi_n(t)]+1} - T_{[n\psi_n(t)]}$$

and

$$P_{\mu_0}\left(\frac{1}{\sqrt{n}} \sup_{0\leqslant k\leqslant [n\psi_n(t)]} (T_{k+1} - T_k) > \varepsilon \right) \leqslant \{[n\psi_n(t)] + 1\}\, P_{\mu_0}\left(\frac{1}{\sqrt{n}} T_1 > \varepsilon \right)$$

by the stationarity of the increments of T_n. From the Markov inequality

$$P_{\mu_0}\left(\frac{1}{\sqrt{n}} T_1 > \varepsilon \right) \leqslant \frac{E_{\mu_0}(T_1^{2+\delta})}{\varepsilon^{2+\delta} n^{1+\delta/2}}$$

and this completes the proof via $E_{\mu_0}(T_1^{2+\delta}) < \infty$ and $\psi_n(t) \to t$. ∎

The IP under the initial condition that the particle starts at a scatterer now becomes:

Lemma 3.4. Assume (C) and $E_{\mu_0} |u(e, \eta)|^{2+\delta} < \infty$ for some $\delta > 0$ and for all $e \in S_d$. Then as $n \to \infty$ the processes $\{n^{-1/2} X_{[nt]}: t \geqslant 0\}$ converge weakly in μ_0-measure to $\{W_{\alpha D}(t): t \geqslant 0\}$, where $\alpha^{-1} = E_{\mu_0}(T_1)$ and D is the diffusion matrix of Theorem 3.1.

Proof. By Lemma 3.3,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{[T_n t]} Y_k \to W_D(t)$$

By Lemma 3.2, this is the same as

$$\frac{1}{\sqrt{T_n}} \sum_{k=1}^{[T_n t]} Y_k \to W_{\alpha D}(t)$$

Apply the random-time-change theorem (ref. 2, Chapter 3, Section 3) to get

$$\frac{1}{\sqrt{n}} X_{[nt]} = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} Y_k \to W_{\alpha D}(t) \quad ∎$$

3.4. Invariance Principle on the Full Time Scale for μ

Under the measure μ_0 the test particle starts on a scatterer. We shall now show that the IP carries over to the measure μ. The proof is based on the following observation: when the particle hits the first scatterer, it sees an environment distributed according to a measure ν on Ω_0 that is absolutely continuous w.r.t. μ_0. The path between the origin and the first scatterer is negligible in the scaling limit, while for the rest of the path we can use Lemma 3.4 starting from the measure ν.

Lemma 3.5. Assume (C). Assume further that $E_{\mu_0}|u(e,\eta)|<\infty$ for all $e\in S_d$. Let ν_e be the distribution of $\tau_{u(e,\eta)}\eta$ when η is distributed according to μ. Then ν_e (viewed as a measure on Ω_0) is absolutely continuous w.r.t. μ_0, with

$$\frac{d\nu_e}{d\mu_0}(\eta)=|u(-e,\eta)|\,\mu(\eta(0)=1)$$

Proof. Straightforward. ∎

We now obtain our final result.

Theorem 3.6 (*Invariance principle for the stochastic Lorentz lattice gas*). Assume (C) and $E_{\mu_0}|u(e,\eta)|^{2+\delta}<\infty$ for some $\delta>0$ and for all $e\in S_d$. Then as $n\to\infty$ the processes $\{n^{-1/2}X_{[nt]}: t\geqslant 0\}$ converge weakly in μ-measure to $\{W_{\alpha D}(t): t\geqslant 0\}$, where $\alpha^{-1}=E_{\mu_0}(T_1)$ and D is the diffusion matrix of Theorem 3.1.

Proof. Let $X_n(\eta)$ denote the position of the particle at time n in the scatterer configuration η, and assume that the particle starts with velocity e. From the identity

$$X_n(\eta)=(|u(e,\eta)|\wedge n)e+X_{(|u(e,\eta)|\vee n)-|u(e,\eta)|}(\tau_{u(e,\eta)}\eta)$$

it follows that the scaling behavior is determined by

$$\frac{1}{\sqrt{n}}X_{(|u(e,\eta)|\vee[nt])-|u(e,\eta)|}(\tau_{u(e,\eta)}\eta)$$

Now use Lemmas 3.4 and 3.5 and the following observation. If the processes $\{\varepsilon X_{\varepsilon^{-2}t}: t\geqslant 0\}$ converge weakly to a Wiener process as ε tends to zero, then the same is true for $\{\varepsilon X_{(\gamma\vee\varepsilon^{-2}t)-\gamma}: t\geqslant 0\}$ for any $\gamma>0$. ∎

4. NONDEGENERACY OF THE DIFFUSION MATRIX

In this section we give a simple computation of the diffusion constant in $d=1$ and prove the nondegeneracy of the diffusion matrix in $d\geqslant 2$. For a related Lorentz model and under stronger restrictions the value in $d=1$ was computed in refs. 3 and 4 via Green's function techniques.

In $d=1$ the set of scatterers in a configuration $\eta \in \Omega_0$ can be indexed by the integers in the following natural way: $\eta^{-1}\{1\} = \{X_x(\eta): x \in \mathbb{Z}\}$, with $X_0(\eta)=0$ and $X_x(\eta) < X_y(\eta)$ iff $x<y$.

The interscatterer distances are $\delta_x(\eta) = X_{x+1}(\eta) - X_x(\eta)$. In this section expectation w.r.t. μ_0 will be abbreviated by $\langle \cdot \rangle$.

Theorem 4.1. Assume (C) and (i) $\langle \delta_0^2 \rangle < \infty$; (ii) $\sum_{x \in \mathbb{Z}} |\langle \delta_0 \delta_x \rangle - \langle \delta_0 \rangle \langle \delta_x \rangle| < \infty$. Then $\alpha D = \langle \delta_0 \rangle$.

Proof. Observe that the process $\{X_{T_n}: n \geq 0\}$ is a simple random walk on the set $\{X_x(\eta): x \in \mathbb{Z}\}$. Therefore [recall (3.2)]

$$P'^n \delta_x(\eta) = \sum_{y \in \mathbb{Z}} p_n(x, y)\, \delta_y(\eta) \qquad \text{for all} \quad n \geq 0, \quad x \in \mathbb{Z}, \quad \eta \in \Omega_0 \tag{4.1}$$

where $p_n(x, y)$ denotes the n-step transition probability to go from x to y for the simple random walk on $\mathbb{Z}$. This formula can be used to compute the dynamic part of the diffusion constant [recall (3.3)] as follows:

$$\begin{aligned} D^d &= \frac{1}{2} \lim_{N \to \infty} \sum_{n=0}^{N} \langle (\delta_0 - \delta_{-1})\, P'^n (\delta_0 - \delta_{-1}) \rangle \\ &= \frac{1}{2} \lim_{N \to \infty} \sum_{n=0}^{N} \sum_{y \in \mathbb{Z}} [p_n(0, y) - p_n(-1, y)] \langle (\delta_0 - \delta_{-1}) \delta_y \rangle \\ &= \frac{1}{2} \sum_{y \in \mathbb{Z}} \lim_{N \to \infty} \sum_{n=0}^{N} [p_n(0, y) - p_n(0, y+1)] \langle (\delta_0 - \delta_{-1}) \delta_y \rangle \end{aligned} \tag{4.2}$$

The interchange of the sum over y and the limit $N \to \infty$ is allowed because assumption (ii) implies $\sum_{y \in \mathbb{Z}} |\langle (\delta_0 - \delta_{-1}) \delta_y \rangle| < \infty$. The sum over n can be expressed in terms of the potential kernel for simple random walk (ref. 5, Section 29):

$$\lim_{N \to \infty} \sum_{n=0}^{N} [p_n(0, 0) - p_n(0, x)] = |x|$$

This gives

$$\begin{aligned} D^d &= \frac{1}{2} \sum_{y \in \mathbb{Z}} (-|y| + |y+1|) \langle (\delta_0 - \delta_{-1}) \delta_y \rangle \\ &= \langle \delta_0^2 \rangle - \langle \delta_0 \rangle^2 \end{aligned}$$

The static part is $D^s = \langle \delta_0^2 \rangle$, and $\alpha^{-1} = \langle \delta_0 \rangle$. Therefore we end up with

$$\alpha D = \frac{D^s - D^d}{\langle \delta_0 \rangle} = \langle \delta_0 \rangle \quad \blacksquare$$

In $d \geqslant 2$ the diffusion matrix is not explicitly calculable, but we can still prove nondegeneracy.

Theorem 4.2. Assume (C) and assume (i) and (ii) of Theorem 4.1 for the one-dimensional marginals of μ_0. Then

$$\frac{1}{d} \langle |u(e_i, \eta)| \rangle^2 \leqslant D_{ii} = D_i^s - D_i^d \leqslant \frac{1}{d} \langle |u(e_i, \eta)|^2 \rangle$$

Proof. The statement in the theorem is equivalent to [recall (3.3)]

$$0 \leqslant D_i^d \leqslant \frac{1}{d} [\langle |u(e_i, \eta)|^2 \rangle - \langle |u(e_i, \eta)| \rangle^2]$$

The lower bound is a trivial consequence of reversibility. We prove the upper bound as follows. The generator of the EP is of the type

$$L = \frac{1}{d} \sum_{j=1}^{d} L_j = P' - I = \frac{1}{d} \sum_{j=1}^{d} (P_j' - I)$$

where $L_j = P_j' - I$ is the generator for jumps in the jth coordinate direction. Therefore

$$\begin{aligned} D_i^d &= 2 \left\langle \Phi_i \left[\frac{1}{d} \sum_{j=1}^{d} (-L_j) \right]^{-1} \Phi_i \right\rangle \\ &\leqslant 2d \langle \Phi_i (-L_i)^{-1} \Phi_i \rangle \\ &= \frac{1}{d} [\langle |u(e_i, \eta)|^2 \rangle - \langle |u(e_i, \eta)| \rangle^2] \end{aligned}$$

The inequality comes from the fact that each $-L_j$ is a positive self-adjoint operator. This is explained in detail in ref. 1, Section 3. The last equality uses (3.3) and the result for D^d in the proof of Theorem 4.1. $\blacksquare$

Remark. If assumption (ii) does not hold for the one-dimensional marginals of μ_0, then we still have nondegeneracy under conditions (C) and (i), because in that case we can use the trivial lower bound $1/d \leqslant D_{ii}$. This bound is obtained by noting that $D \geqslant 1$ in $d = 1$, and that D is an increasing function of the dimension (ref. 1, Section 3).

5. RECURRENCE IN $d=2$

Recall that P^η denotes the measure on path space induced by the process $\{X_{T_n}: n \geqslant 0\}$ starting at 0 in the scatterer configuration η ($\eta \in \Omega_0$). Let μ be a probability measure on Ω that is stationary under translations, such that $\mu(\eta(0)=1)>0$. Let $\chi(x, y; \eta)$ denote the indicator of the event that x, y are neighboring scatterers in the configuration η. Then the transition matrix for the random walk on the scatterers of η is

$$p^\eta(x, y) = \frac{1}{2d}\chi(x, y; \eta)$$

We shall make a comparison with the random walk on $\mathbb{Z}^d$ whose transition matrix is given by

$$p_{\mu_0}(x, y) = \int p^\eta(0, y-x)\, \mu_0(d\eta)$$

This random walk will be denoted by $\{\tilde{X}_n: n \geqslant 0\}$ and will be called *the effective medium random walk*. The measure on path space induced by this process starting at 0 will be denoted by P^{eff}.

For $\Lambda \subset \mathbb{Z}^d$, $0 \in \Lambda$, $|\Lambda| < \infty$, let

$$\begin{aligned}
\tau_\Lambda &= \inf\{n \geqslant 0: X_{T_n} \notin \Lambda\}\\
\tau_0 &= \inf\{n \geqslant 1: X_{T_n} = 0\}\\
\tau_0^{\mathrm{eff}} &= \inf\{n \geqslant 1: \tilde{X}_n = 0\}
\end{aligned}$$

and

$$H(\Lambda) = \{h: \mathbb{Z}^d \to [0, 1]: h(0)=1,\ h\,|\,\Lambda^c \equiv 0\}$$

The following theorem is in the spirit of Durrett.[6]

Theorem 5.1. Suppose that

(i) μ is stationary under translations.

(ii) $\mu(\eta(0)=1)>0$.

(iii) μ is connected.

Then

$$\int [1 - P^\eta(\tau_0 < \infty)]\, d\mu_0(\eta) \leqslant 1 - P^{\mathrm{eff}}(\tau_0^{\mathrm{eff}} < \infty) \tag{5.1}$$

Proof. First note that

$$1 - P^\eta(\tau_0 < \infty) = \lim_{\Lambda \uparrow Z^d} P^\eta(\tau_\Lambda \leqslant \tau_0)$$

By the Dirichlet principle (see, e.g., ref. 7, Theorem II. 6.1)

$$P^\eta(\tau_\Lambda \leqslant \tau_0) = \inf_{h \in H(\Lambda)} E(\Lambda, h, \eta)$$

with

$$E(\Lambda, h, \eta) = \frac{1}{2} \sum_{x, y} p^\eta(x, y)[h(y) - h(x)]^2$$

Therefore

$$\begin{aligned}
&\int [1 - P^\eta(\tau_0 < \infty)]\, d\mu_0(\eta) \\
&\quad = \int [\lim_{\Lambda \uparrow Z^d} \inf_{h \in H(\Lambda)} E(\Lambda, h, \eta)]\, d\mu_0(\eta) \\
&\quad \leqslant \lim_{\Lambda \uparrow Z^d} \inf_{h \in H(\Lambda)} \int E(\Lambda, h, \eta)\, d\mu_0(\eta) \\
&\quad \leqslant \lim_{\Lambda \uparrow Z^d} \inf_{h \in H(\Lambda)} \frac{1}{2} \sum_{x, y} p_{\mu_0}(x, y)[h(y) - h(x)]^2 \\
&\quad = 1 - P^{\mathrm{eff}}(\tau_0^{\mathrm{eff}} < \infty)
\end{aligned}$$

where in the last step we use the Dirichlet principle for the effective medium random walk. ∎

Remark. Assumption (iii) ensures the irreducibility of the random walk on the scatterers of η, which is needed for the Dirichlet principle.

Theorem 5.2 (Recurrence in $d \leqslant 2$). Suppose that μ is a probability measure on Ω satisfying:

(i) μ is stationary under translations.

(ii) $E_{\mu_0} |u(e, \eta)|^2 < \infty$ for all $e \in S_d$.

(iii) μ is connected.

Then in $d \leqslant 2$

$$P^\eta(X_{T_n} = 0 \text{ infinitely often}) = 1 \qquad \mu_0\text{-a.s.}$$

Proof. In $d=1$ the statement of the theorem is trivial. So consider $d=2$. By assumptions (i) and (ii) the effective medium random walk is symmetric and has finite variance and therefore is recurrent. Hence the r.h.s. of (5.1) is zero. Therefore $P^{\eta}(\tau_0 < \infty) = 1$ μ_0-a.s. ∎

REFERENCES

1. A. DeMasi, P. A. Ferrari, S. Goldstein, and W. D. Wick, An invariance principle for reversible Markov processes. Applications to random motions in random environments, *J. Stat. Phys.* **55**:787–855 (1989).
2. P. Billingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).
3. H. van Beijeren, Transport properties of stochastic Lorentz models, *Rev. Mod. Phys.* **54**:195–234 (1982).
4. H. van Beijeren and H. Spohn, Transport properties of the one dimensional stochastic Lorentz model. I. velocity autocorrelation, *J. Stat. Phys.* **31**:231–254 (1982).
5. F. Spitzer, *Principles of Random Walk* (Van Nostrand, Princeton, New Jersey, 1964).
6. R. Durrett, Multidimensional random walks in random environments with subclassical limiting behavior, *Commun. Math. Phys.* **104**:87–102 (1986).
7. T. M. Liggett, *Interacting Particle Systems* (Springer, Berlin, 1985).
8. M. Rosenblatt, Transition probability operators, *Proc. Fifth Berkeley Symp. Math. Stat. Prob.* **2**:473–483 (1967).
9. S. R. S. Varadhan, in: *Theory and Application of Random Fields*, G. Kallianpur, ed. (Springer-Verlag, Berlin, 1983), pp. 282–283.